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A new 'doubly special relativity' theory from a quantum Weyl–Poincaré algebra

Ángel Ballesteros[1], **N Rossano Bruno**[1,2] **and Francisco J Herranz**[1]

[1] Departamento de Física, Universidad de Burgos, Pza Misael Bañuelos s.n., 09001 Burgos, Spain
[2] Dipartimento di Fisica, Università di Roma Tre and INFN Sez. Roma Tre, Via Vasca Navale 84, 00146 Roma, Italy

E-mail: angelb@ubu.es, rossano@fis.uniroma3.it and fjherranz@ubu.es

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Abstract
A 'mass-like' quantum Weyl–Poincaré algebra is proposed to describe, after the identification of the deformation parameter with the Planck length, a new relativistic theory with two observer-independent scales (or 'doubly special relativity' (DSR) theory). Deformed momentum representation, finite boost transformations, range of rapidity, energy and momentum, and position and velocity operators are explicitly studied and compared with those of previous DSR theories based on κ-Poincaré algebra. The main novelties of the DSR theory presented here are the new features of momentum saturation and a new type of deformed position operators.

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1. Introduction

In the last few years several approaches to the problem of unification of general relativity and quantum mechanics have led to arguments in favour of the modification of Lorentz symmetry at the Planck scale [1]. The different approaches to quantum gravity such as loop quantum gravity [2, 3], or string theory [4, 5] assign to the Planck scale a fundamental role in the structure of spacetime and of momentum space, or as linked to discrete spectra of physical observables [6].

The so-called doubly special relativity (DSR) theories (see [7] and references therein) have been proposed as possible tools to investigate this ongoing quantum gravity debate in order to reconcile Lorentz invariance with the new fundamental role assigned to the Planck scale. Different DSR proposals [8–10] introduce, in addition to the usual observer-independent velocity scale, an observer-independent length scale (momentum scale), possibly related to the Planck scale ($L_{\rm p} \simeq 10^{-33}$ cm), which acquires the role of a minimum length (maximum momentum) [9, 11], in a way in which Lorentz invariance is preserved. Also the observable implications of DSR theories are being studied in connection with some planned Lorentz-symmetry tests [12, 13], and in searching for a kinematical solution for the puzzling observations of ultra-high-energy cosmic rays [14–16].

Under investigation [11, 17, 18] is the role that quantum groups can play in these DSR theories; for most of them, the κ-Poincaré algebra [19–21] seems to play a role which is analogous to that played by the Poincaré group in Einstein's special relativity. However, in pursuing this connection between relativistic theories with two observer-independent scales and quantum groups, it is natural to consider that *any* quantum Poincaré algebra should be taken as a first stage prior to the search for more general DSR theories which should be endowed with either a quantum conformal symmetry or a q-AdS (q-dS) symmetry. In this respect, several quantum deformations for $so(p, q)$ algebras of non-standard or twisted type (different from the Drinfeld–Jimbo deformation) have been constructed during the last few years. Some of them have been obtained within a conformal framework as quantum conformal algebras of the Minkowskian spacetime such as $so(2, 2)$ [22, 23], $so(3, 2)$ [24], and more recently $so(4, 2)$ [25–28]. From a more general mathematical perspective, twisted $so(p, q)$ algebras have been deduced by applying different types of Drinfeld's twists [29, 30] but most of them are written in terms of Cartan–Weyl bases.

In spite of all the mathematical (Hopf structures) background so obtained, most of the possible physical applications have mainly been studied in relation to difference–differential symmetries on uniform discretizations of the Minkowskian spacetime along a certain direction, for which the deformation parameter plays the role of the lattice step [23, 25, 27]. The aim of this paper is to explore and extract the physical consequences provided by a new DSR proposal by considering the most manageable deformation of the Weyl–Poincaré algebra, $U_\tau(\mathcal{WP})$, which arises within a recently introduced quantum $so(4, 2)$ algebra [25]. The Hopf structure and the deformed momentum representation for $U_\tau(\mathcal{WP})$ are presented in the next section. Afterwards, by means of the usual techniques developed in studying previous DSR theories, we deduce the physical implications. Among them, finite boost transformations are obtained in section 3, and a detailed analysis of the range of the boost parameter is performed in section 4. We find that the range of the rapidity depends on the sign of the deformation parameter and, in general, the behaviour of the rapidity, energy and momentum differs from DSR theories based on κ-Poincaré. For a positive deformation parameter, the situation is rather similar to the undeformed case, while for a negative value, the range of the rapidity is restricted between two asymptotic values for the energy; furthermore such values determine a maximum momentum that depends not only on the deformation parameter (as in κ-Poincaré) but also on the deformed mass of the particle. In section 5, two proposals for position and velocity operators are presented: one of them provides a variable speed of light for massless particles, while the other one gives rise to a fixed speed of light and conveys a new type of generalized uncertainty principle as well. Some conclusions and remarks close the paper.

2. Quantum Weyl–Poincaré algebra

Let us consider the mass-like quantum conformal Minkowskian algebra introduced in [25], $U_\tau(so(4, 2))$, where τ is the deformation parameter. The ten Poincaré generators together with the dilation close a Weyl–Poincaré (or similitude) Hopf subalgebra, $U_\tau(\mathcal{WP}) \subset U_\tau(so(4, 2))$. Such a deformation is the natural extension to $3+1$ dimensions of the results previously presented in [23] for the time-type quantum $so(2, 2)$ algebra and, in fact, is based on the Jordanian twist (introduced in [31]) that underlies the non-standard (or h-deformation)

$U_\tau(sl(2,\mathbb{R}))$ [32]. Thus $U_\tau(so(4,2))$ verifies the following sequence of Hopf subalgebra embeddings:

$$\begin{array}{ccccc} U_\tau(sl(2,\mathbb{R})) \simeq U_\tau(so(2,1)) \subset U_\tau(so(2,2)) & \subset & U_\tau(so(3,2)) & \subset & U_\tau(so(4,2)) \\ \cup & & \cup & & \cup \\ U_\tau(\mathcal{WP}_{1+1}) & \subset & U_\tau(\mathcal{WP}_{2+1}) & \subset & U_\tau(\mathcal{WP}_{3+1}). \end{array}$$

This, in turn, ensures that properties associated with deformations in low dimensions are fulfilled, by construction, in higher dimensions and moreover any physical consequence derived from the structure of $U_\tau(\mathcal{WP})$ is consistent with a full quantum conformal symmetry that can be further developed. Therefore, throughout the paper we will restrict ourselves to analysing the DSR theory provided by the deformed Weyl–Poincaré symmetries after the identification of the deformation parameter with the Planck length: $\tau \sim L_{\mathrm{p}}$.

If $\{J_i, P_\mu = (P_0, \mathbf{P}), K_i, D\}$ denote the generators of rotations, time and space translations, boosts and dilations, the non-vanishing deformed commutation rules and coproduct of $U_\tau(\mathcal{WP})$ are given by [25]:

$$\begin{array}{lll} [J_i, J_j] = \mathrm{i}\,\varepsilon_{ijk} J_k & [J_i, K_j] = \mathrm{i}\,\varepsilon_{ijk} K_k & [J_i, P_j] = \mathrm{i}\,\varepsilon_{ijk} P_k \\ [K_i, K_j] = -\mathrm{i}\,\varepsilon_{ijk} J_k & [K_i, P_0] = \mathrm{i}\,\mathrm{e}^{-\tau P_0} P_i & [D, P_i] = \mathrm{i}\, P_i \\ [K_i, P_i] = \mathrm{i}\,\dfrac{\mathrm{e}^{\tau P_0} - 1}{\tau} & [D, P_0] = \mathrm{i}\,\dfrac{1 - \mathrm{e}^{-\tau P_0}}{\tau} & \end{array} \tag{2.1}$$

$$\begin{array}{ll} \Delta(P_0) = 1 \otimes P_0 + P_0 \otimes 1 & \Delta(P_i) = 1 \otimes P_i + P_i \otimes \mathrm{e}^{\tau P_0} \\ \Delta(J_i) = 1 \otimes J_i + J_i \otimes 1 & \Delta(D) = 1 \otimes D + D \otimes \mathrm{e}^{-\tau P_0} \\ \Delta(K_i) = 1 \otimes K_i + K_i \otimes 1 - \tau D \otimes \mathrm{e}^{-\tau P_0} P_i & \end{array} \tag{2.2}$$

where hereafter we assume $\hbar = c = 1$, sum over repeated indices, Latin indices $i, j, k = 1, 2, 3$, while Greek indices $\mu, \nu = 0, 1, 2, 3$. The generators of $U_\tau(\mathcal{WP})$ are Hermitian operators and τ is a *real* deformation parameter.

Counit ϵ and antipode S can be directly deduced from (2.1) and (2.2); they read

$$\begin{array}{lll} \epsilon(X) = 0 & X \in \{J_i, P_\mu, K_i, D\} & \epsilon(1) = 1 \\ S(P_0) = -P_0 & S(P_i) = -P_i\,\mathrm{e}^{-\tau P_0} & S(J_i) = -J_i \\ S(K_i) = -K_i - \tau D P_i & S(D) = -D\,\mathrm{e}^{\tau P_0} & S(1) = 1. \end{array} \tag{2.3}$$

The Poincaré sector of $U_\tau(\mathcal{WP})$ (which does not close a Hopf subalgebra due to the coproduct of the boosts (2.2)) provides one useful operator. If P_0 is considered as the energy of a particle, the deformation of the quadratic Poincaré Casimir

$$M^2 = \left(\frac{\mathrm{e}^{\tau P_0} - 1}{\tau}\right)^2 - \mathbf{P}^2 \tag{2.4}$$

can be assumed as the deformed mass-shell condition related to the rest mass m by

$$m = \frac{1}{\tau}\ln(1 + \tau M) \qquad \lim_{\tau\to 0} M = m. \tag{2.5}$$

Alternatively, the deformed Poincaré Casimir (2.4) has been used to obtain a time discretization of the wave or massless Klein–Gordon equation with quantum conformal symmetry once τ is identified with the time lattice constant [23, 25].

The operator M allows us to introduce the following deformed momentum representation for the Poincaré generators of (2.1) in terms of $\mathbf{p} = (p^1, p^2, p^3)$ for a spinning massive particle:

$$P_0 = p^0 = \frac{1}{z}\ln\left(1 + z\sqrt{M^2 + \mathbf{p}^2}\right) \qquad J_i = \mathrm{i}\,\varepsilon_{ijk}\,p^k\,\frac{\partial}{\partial p^j} + S_i$$
$$\mathbf{P} = \mathbf{p} \qquad K_i = \mathrm{i}\,\sqrt{M^2 + \mathbf{p}^2}\,\frac{\partial}{\partial p^i} + \varepsilon_{ijk}\,\frac{p^j S_k}{M + \sqrt{M^2 + \mathbf{p}^2}} \tag{2.6}$$

provided that the components of the spin $\mathbf{S}$ fulfil $[S_i, S_j] = \mathrm{i}\,\varepsilon_{ijk} S_k$.

3. Deformed finite boost transformations

The explicit form of the commutation rules $[K_i, P_\mu]$ in (2.1) shows that the action of the boost generators on momentum space is deformed, so we can expect that the associated finite boost transformations are also deformed similarly to the κ-Poincaré case [11, 18, 33].

By taking into account that the Hopf algebra $U_\tau(\mathcal{WP})$ (2.1)–(2.3) resembles a bicrossproduct structure, we introduce the corresponding quantum adjoint action [34] as

$$\mathrm{ad}_Y X = -\sum_i S\big(Y_i^{(1)}\big) X Y_i^{(2)} \tag{3.1}$$

provided that the coproduct of Y is written in Sweedler's notation as $\Delta(Y) = \sum_i Y_i^{(1)} \otimes Y_i^{(2)}$ and S is the antipode (2.3).

By using (3.1) it can be checked that

$$\mathrm{ad}_{K_i}\mathcal{F}(P_\mu) = [K_i, \mathcal{F}(P_\mu)] \tag{3.2}$$

for any momentum-dependent smooth function $\mathcal{F}$. Next, we consider a boost transformation along a generic direction determined by a unitary vector $\mathbf{u}$. If $P_\mu^0 = \big(P_0^0, \mathbf{P}^0\big)$ are the measurements performed by the first observer with rapidity $\xi = 0$ and $P_\mu = (P_0, \mathbf{P})$ the measurements performed by the second one with arbitrary ξ, the finite boost transformation associated with $\mathbf{u}\cdot\mathbf{K}$ is obtained from (3.2) and reads

$$P_\mu = \sum_{n=0}^{\infty} \frac{1}{n!}\mathrm{ad}^{n)}_{-\mathrm{i}\xi\mathbf{u}\cdot\mathbf{K}} P_\mu^0 = \exp\{-\mathrm{i}\xi\mathbf{u}\cdot\mathbf{K}\} P_\mu^0 \exp\{\mathrm{i}\xi\mathbf{u}\cdot\mathbf{K}\}. \tag{3.3}$$

Hence the infinitesimal boost transformation is given by

$$\frac{\mathrm{d}P_\mu}{\mathrm{d}\xi} = -\mathrm{i}\,[\mathbf{u}\cdot\mathbf{K}, P_\mu]. \tag{3.4}$$

By substituting (2.1) into (3.4) we obtain a system of coupled differential equations:

$$\frac{\mathrm{d}P_0}{\mathrm{d}\xi} = \mathrm{e}^{-\tau P_0}\mathbf{u}\cdot\mathbf{P} \qquad \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\xi} = \mathbf{u}\left(\frac{\mathrm{e}^{\tau P_0} - 1}{\tau}\right) \tag{3.5}$$

which give rise to a unique non-linear differential equation for P_0:

$$\frac{\mathrm{d}^2 P_0}{\mathrm{d}\xi^2} + \tau\left(\frac{\mathrm{d}P_0}{\mathrm{d}\xi}\right)^2 + \frac{\mathrm{e}^{-\tau P_0} - 1}{\tau} = 0. \tag{3.6}$$

Therefore we obtain that

$$P_0(\xi) = \frac{1}{\tau}\ln\left(\frac{2 + a_+\mathrm{e}^{\xi} + a_-\mathrm{e}^{-\xi}}{2}\right) \qquad \mathbf{P}(\xi) = \frac{\mathbf{u}}{2\tau}(a_+\mathrm{e}^{\xi} - a_-\mathrm{e}^{-\xi}) + \mathbf{a} \tag{3.7}$$

where $a_\pm$ and $\mathbf{a}$ are integration constants. By imposing the initial conditions $P_\mu(0) = P_\mu^0$, we find that

$$a_\pm = \pm\tau\mathbf{u}\cdot\mathbf{P}^0 + \big(\mathrm{e}^{\tau P_0^0} - 1\big) \qquad \mathbf{a} = \mathbf{P}^0 - \mathbf{u}(\mathbf{u}\cdot\mathbf{P}^0) \tag{3.8}$$

so that the deformed finite boost transformations (3.3) turn out to be

$$
\begin{aligned}
P_0(\xi) &= \frac{1}{\tau}\ln\left(1+\left(\mathrm{e}^{\tau P_0^0}-1\right)\cosh\xi+\tau\mathbf{u}\cdot\mathbf{P}^0\sinh\xi\right)\\
\mathbf{P}(\xi) &= \mathbf{P}^0+\mathbf{u}\left(\mathbf{u}\cdot\mathbf{P}^0(\cosh\xi-1)+\frac{1}{\tau}\left(\mathrm{e}^{\tau P_0^0}-1\right)\sinh\xi\right).
\end{aligned}
\tag{3.9}
$$

It can be checked that the deformed mass-shell condition (2.4) remains invariant under (3.9). Furthermore, this result allows us to deduce the composition of deformed boost transformations along two directions characterized by the unitary vectors $\mathbf{u}_1, \mathbf{u}_2$ such that $\mathbf{u}_1\cdot\mathbf{u}_2=\cos\theta$. Starting from the initial observer with P_μ^0, we consider a first transformation $\mathbf{u}_1\cdot\mathbf{K}$ with rapidity ξ_1 to a second observer which measures P_μ, and next a second transformation $\mathbf{u}_2\cdot\mathbf{K}$ with boost parameter ξ_2 to a third observer which measures P'_μ:

$$
P_\mu^0 \overset{\xi_1,\mathbf{u}_1\cdot\mathbf{K}}{\longrightarrow} P_\mu = P_\mu\left(\xi_1; P_\mu^0\right) \overset{\xi_2,\mathbf{u}_2\cdot\mathbf{K}}{\longrightarrow} P'_\mu = P'_\mu(\xi_2; P_\mu) = P'_\mu\left(\xi_1,\xi_2; P_\mu^0\right).
$$

Due to expressions (3.2) and (3.3), this sequence corresponds to

$$
P'_\mu = \exp\{-\mathrm{i}\,\xi_2\mathbf{u}_2\cdot\mathbf{K}\}\exp\{-\mathrm{i}\,\xi_1\mathbf{u}_1\cdot\mathbf{K}\}P_\mu^0\exp\{\mathrm{i}\,\xi_1\mathbf{u}_1\cdot\mathbf{K}\}\exp\{\mathrm{i}\,\xi_2\mathbf{u}_2\cdot\mathbf{K}\} \tag{3.10}
$$

which coincides with the classical Lie group expression. Nevertheless, we stress that this is a consequence of the complete Hopf structure underlying the definition of the quantum adjoint action (3.1).

The resulting composition is given by

$$
\begin{aligned}
P'_0 = \frac{1}{\tau}\ln\Big\{&1+\left(\mathrm{e}^{\tau P_0^0}-1\right)(\cosh\xi_1\cosh\xi_2+\sinh\xi_1\sinh\xi_2\cos\theta)\\
&+\tau\mathbf{u}_1\cdot\mathbf{P}^0(\sinh\xi_1\cosh\xi_2+\cosh\xi_1\sinh\xi_2\cos\theta)-\tau(\mathbf{u}_1\cos\theta-\mathbf{u}_2)\cdot\mathbf{P}^0\sinh\xi_2\Big\}\\
\mathbf{P}' = \mathbf{P}^0 &+ \mathbf{u}_2\{\mathbf{u}_1\cdot\mathbf{P}^0(\cosh\xi_1\cosh\xi_2\cos\theta+\sinh\xi_1\sinh\xi_2)-\mathbf{u}_2\cdot\mathbf{P}^0\}\\
&+\frac{1}{\tau}\left(\mathrm{e}^{\tau P_0^0}-1\right)\{\mathbf{u}_2(\sinh\xi_1\cosh\xi_2\cos\theta+\cosh\xi_1\sinh\xi_2)+\mathbf{n}\sinh\xi_1\}\\
&+\mathbf{n}(\mathbf{u}_1\cdot\mathbf{P}^0)(\cosh\xi_1-1)-\mathbf{u}_2(\mathbf{n}\cdot\mathbf{P}^0)\cosh\xi_2
\end{aligned}
\tag{3.11}
$$

where we have introduced the shorthand notation $\mathbf{n}=\mathbf{u}_1-\mathbf{u}_2\cos\theta$.

As a straightforward consequence, if both deformed boost transformations are performed along the same direction $\mathbf{u}_1=\mathbf{u}_2$, then $\theta=0$ and $\mathbf{n}=0$, so that $P'_\mu=P'_\mu\left(\xi_1+\xi_2; P_\mu^0\right)$, and thus we obtain the additivity of the rapidity in the same way as for κ-Poincaré [33].

4. Range of rapidity, energy and momentum

The stationary points of the energy can be studied by means of the derivatives of $P_0(\xi)$. For the sake of simplicity we shall analyse the (1 + 1)-dimensional case.

4.1. Massive particles

Let us first consider massive particles. If we particularize the transformations (3.9) to $P_\mu=(P_0,P_1)$, drop the index 0 in P_μ^0 and introduce the deformed mass (2.4), we find that the deformed finite boost transformations can be rewritten as

$$
\begin{aligned}
P_0(\xi) &= \frac{1}{\tau}\ln\left(1+\tau\sqrt{M^2+P_1^2}\cosh\xi+\tau P_1\sinh\xi\right)\\
P_1(\xi) &= P_1\cosh\xi+\sqrt{M^2+P_1^2}\sinh\xi.
\end{aligned}
\tag{4.1}
$$

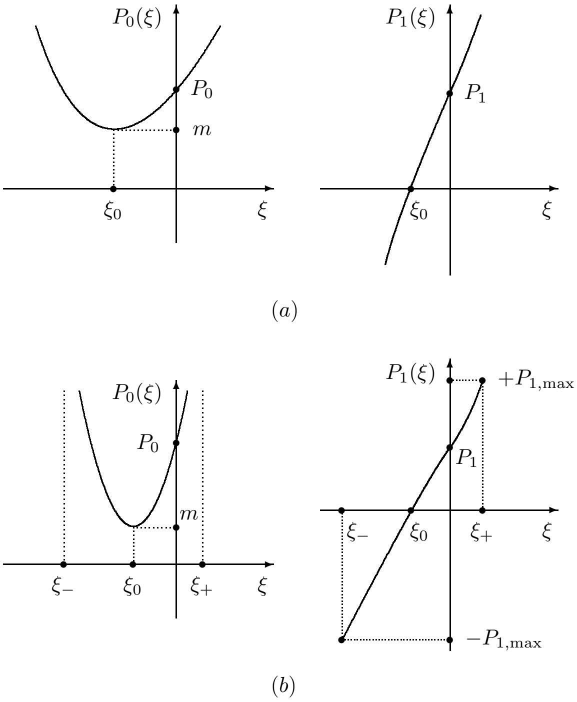

Figure 1. Energy $P_0(\xi)$ and momentum $P_1(\xi)$ for a massive particle with a positive initial momentum P_1 according to the sign of τ: (*a*) positive deformation parameter τ and (*b*) $\tau < 0$, $M < 1/|\tau|$ and $P_{1,\max} = \sqrt{1 - \tau^2 M^2}/|\tau|$.

The zero value for the first derivative of $P_0(\xi)$ gives the following expression for the rapidity:

$$\tanh \xi_0 = -\frac{P_1}{\sqrt{M^2 + P_1^2}} \qquad \xi_0 = \ln\left(\frac{M}{\sqrt{M^2 + P_1^2} + P_1}\right) \tag{4.2}$$

for which the energy takes the value of the physical rest mass of the particle under consideration and the momentum vanishes:

$$P_0(\xi_0) = \frac{1}{\tau}\ln(1 + \tau M) = m \qquad P_1(\xi_0) = 0. \tag{4.3}$$

To establish whether this situation corresponds to a minimum of the energy, as it should, we compute the second derivative of $P_0(\xi)$. Thus we obtain

$$\left.\frac{\mathrm{d}^2 P_0(\xi)}{\mathrm{d}\xi^2}\right|_{\xi=\xi_0} = \frac{M}{1 + \tau M}. \tag{4.4}$$

Therefore two different situations arise according to the sign of the deformation parameter:

- If $\tau > 0$, ξ_0 always determines a minimum of the energy.
- If $\tau < 0$, ξ_0 provides a minimum only if $M < 1/|\tau|$.

Now, by taking into account these cases we analyse the range of the boost parameter, energy and momentum. When τ is positive the rapidity ξ can take any real value and expressions (4.1) show that both $P_0(\xi)$ and $P_1(\xi)$ are always well defined as well as unbounded (see figure 1(*a*)). In particular, in the limit $\xi \to +\infty$, we find that $P_0(\xi) \to +\infty$, $P_1(\xi) \to +\infty$, while if $\xi \to -\infty$, then $P_0(\xi) \to +\infty$, $P_1(\xi) \to -\infty$, in the same way as in the undeformed case.

In contrast, when τ is negative the condition $M < 1/|\tau|$ must be fulfilled. Then we rewrite the energy as

$$P_0(\xi) = \frac{1}{|\tau|} \ln \left(\frac{1}{1 - |\tau|\sqrt{M^2 + P_1^2} \cosh \xi - |\tau| P_1 \sinh \xi} \right) \tag{4.5}$$

which is always a well-defined expression. However, there exist two values of the boost parameter which give asymptotic values for the energy, namely

$$\xi_- = \ln \left(\frac{1 - \sqrt{1 - \tau^2 M^2}}{|\tau|\big(\sqrt{M^2 + P_1^2} + P_1\big)} \right) \qquad \xi_+ = \ln \left(\frac{1 + \sqrt{1 - \tau^2 M^2}}{|\tau|\big(\sqrt{M^2 + P_1^2} + P_1\big)} \right) \tag{4.6}$$

which are consistent with the constraint $M < 1/|\tau|$. Hence the range of the boost parameter is not the whole real axis but it does have a limited range: $\xi_- < \xi < \xi_+$. This interval is symmetric with respect to ξ_0 (4.2):

$$\xi_+ - \xi_0 = \xi_0 - \xi_- = \ln \left(\frac{1 + \sqrt{1 - \tau^2 M^2}}{|\tau| M} \right). \tag{4.7}$$

The points $\xi_\pm$ show an unbounded energy but provide a maximum momentum:

$$P_0(\xi_\pm) = +\infty \qquad P_1(\xi_\pm) = \pm \frac{1}{|\tau|} \sqrt{1 - \tau^2 M^2}. \tag{4.8}$$

Consequently, whenever τ is negative and $M < 1/|\tau|$, we find that the behaviour of ξ, $P_0(\xi)$ and $P_1(\xi)$ differs from the classical one as depicted in figure 1(*b*); the momentum saturates in the asymptotic values for the energy with a maximum value which is different for each particle since it depends not only on the deformation parameter but also on the deformed mass.

4.2. Massless particles

For massless particles with $M = m = 0$ we have that $e^{\tau P_0} - 1 = \tau |P_1|$, so that equations (4.2) and (4.3) reduce to

$$\tanh \xi_0 = -P_1/|P_1| \qquad P_0(\xi_0) = 0 \qquad P_1(\xi_0) = 0. \tag{4.9}$$

Although the second derivative (4.4) vanishes, the behaviour of massless particles is again deeply determined by the sign of τ.

Firstly, let us consider $\tau > 0$. If the initial momentum $P_1 > 0$, then $\xi_0 \to -\infty$ and in the limit $\xi \to +\infty$, we find that both $P_0(\xi)$, $P_1(\xi) \to +\infty$. Analogously, if $P_1 < 0$, then $\xi_0 \to +\infty$ and $P_0(\xi) \to +\infty$, $P_1(\xi) \to -\infty$ under the limit $\xi \to -\infty$. This situation is also rather similar to the undeformed case (see figure 2(*a*)).

On the other hand, if $\tau < 0$, there is no additional constraint (the above condition $M = 0 < 1/|\tau|$ is trivially fulfilled). According to the sign of the initial momentum, one of the two previous asymptotes disappears and coincides with ξ_0. In particular, if $P_1 > 0$, then $\xi_0 \equiv \xi_- \to -\infty$, the asymptotic value ξ_+ is left and the momentum saturates as shown in figure 2(*b*):

$$\xi \in (-\infty, \xi_+) \qquad \xi_+ = \ln \left(\frac{1}{|\tau| P_1} \right) \qquad P_0(\xi_+) = +\infty \qquad P_1(\xi_+) = \frac{1}{|\tau|}. \tag{4.10}$$

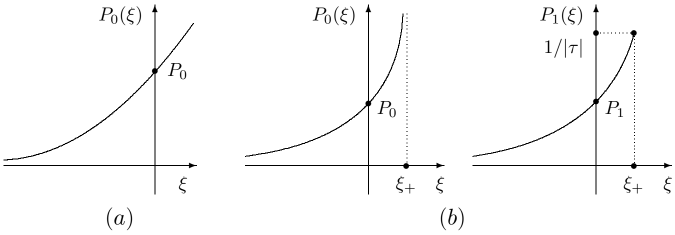

Figure 2. Energy $P_0(\xi)$ and momentum $P_1(\xi)$ for a massless particle with a positive initial momentum P_1 according to the sign of τ: (a) $\tau > 0$; similar for $P_1(\xi)$ and (b) $\tau < 0$.

If $P_1 < 0$, we find that $\xi_0 \equiv \xi_+ \to +\infty$, the rapidity ξ_- is kept and the momentum also saturates:

$$\xi \in (\xi_-, +\infty) \qquad \xi_- = -\ln\left(\frac{1}{|\tau||P_1|}\right) \qquad P_0(\xi_-) = +\infty \qquad P_1(\xi_-) = -\frac{1}{|\tau|}. \quad (4.11)$$

Therefore, the maximum momentum is only determined by the deformation parameter, although the corresponding rapidity depends also on the initial momentum.

5. Position and velocity operators

Position and velocity observables in DSR theories, specially for the κ-Poincaré algebra, have been introduced using different approaches [35–41] and some of them lead to different proposals. In our case, we follow the same algebraic procedure first applied in [36] for κ-Poincaré and also in [37] for the null-plane quantum Poincaré algebra. We consider some generic position Q_i and velocity V_i operators defined as

$$Q_i = \frac{1}{2}\left(\frac{1}{f(\tau,P_0)}K_i + K_i\frac{1}{f(\tau,P_0)}\right) \qquad V_i = \frac{\mathrm{d}Q_i}{\mathrm{d}t} = -\mathrm{i}\,[Q_i, P_0] \quad (5.1)$$

where $f(\tau, P_0)$ is an arbitrary smooth function such that $\lim_{\tau\to 0} f(\tau, P_0) = P_0$. Next we compute the commutation rules between Q_i and the remaining Weyl–Poincaré generators (we omit the arguments in the function f):

$$\begin{aligned} &[J_i, Q_j] = \mathrm{i}\,\varepsilon_{ijk}Q_k \qquad [Q_i, P_0] = \mathrm{i}\,\frac{\mathrm{e}^{-\tau P_0}}{f}P_i \qquad [Q_i, P_j] = \mathrm{i}\,\delta_{ij}\frac{\mathrm{e}^{\tau P_0}-1}{\tau f} \\ &[Q_i, Q_j] = \mathrm{i}\,\frac{f'}{f^2}\mathrm{e}^{-\tau P_0}\left(Q_iP_j - Q_jP_i - \frac{\mathrm{e}^{\tau P_0}}{f'}\varepsilon_{ijk}J_k\right) \\ &[D, Q_i] = -\frac{\mathrm{i}}{2}\left\{\frac{f'}{f^2}\left(\frac{1-\mathrm{e}^{-\tau P_0}}{\tau}\right)K_i + K_i\left(\frac{1-\mathrm{e}^{-\tau P_0}}{\tau}\right)\frac{f'}{f^2}\right\} \end{aligned} \quad (5.2)$$

where f' is the formal derivative of f with respect to P_0. This suggests two natural possibilities for the function f, which are summarized as follows:

(1) $f = (\mathrm{e}^{\tau P_0} - 1)/\tau = \sqrt{M^2 + \mathbf{P}^2}$. Hence we obtain

$$\begin{aligned} &[J_i, Q_j] = \mathrm{i}\,\varepsilon_{ijk}Q_k \qquad [Q_i, P_0] = \mathrm{i}\,\frac{\mathrm{e}^{-\tau P_0}}{\sqrt{M^2+\mathbf{P}^2}}P_i \qquad V_i = \frac{\mathrm{e}^{-\tau P_0}}{\sqrt{M^2+\mathbf{P}^2}}P_i \\ &[D, Q_i] = -\mathrm{i}\,Q_i \qquad [Q_i, Q_j] = -\mathrm{i}\,\varepsilon_{ijk}\frac{\Sigma_k}{M^2+\mathbf{P}^2} \qquad [Q_i, P_j] = \mathrm{i}\,\delta_{ij} \end{aligned} \quad (5.3)$$

where we have introduced the kinematical observables [35, 36]: $\Sigma_k = J_k - \varepsilon_{ijk}Q_iP_j$. By taking into account the representation (2.6), it can be checked that if we consider a spinless massive particle, then $\mathbf{\Sigma} = 0$ (i.e. $\mathbf{J} = \mathbf{Q} \times \mathbf{P}$), so that $[Q_i, Q_j] = 0$, while if we consider a spinning massive particle with $\mathbf{P} = 0$, then $\mathbf{\Sigma} = \mathbf{S}$.

Physical consequences of this choice are directly deduced from the commutation rules (5.3): the position $\mathbf{Q}$ behaves as a classical vector under rotation and dilations, there is no generalized uncertainty principle and the velocity of massless particles is $|\mathbf{V}| = V = \mathrm{e}^{-\tau P_0}$, so that this depends on the energy, which can be either $V = \mathrm{e}^{-\tau P_0} < 1$ for $\tau > 0$, or $V = \mathrm{e}^{|\tau| P_0} > 1$ for $\tau < 0$. This fact is well known in DSR theories based on κ-Poincaré [35, 42].

(2) $f = (1 - \mathrm{e}^{-\tau P_0})/\tau = \mathrm{e}^{-\tau P_0}\sqrt{M^2 + \mathbf{P}^2}$. In this case, we find that

$$[J_i, Q_j] = \mathrm{i}\,\varepsilon_{ijk} Q_k \qquad [Q_i, P_0] = \mathrm{i}\,\frac{P_i}{\sqrt{M^2 + \mathbf{P}^2}} \qquad V_i = \frac{P_i}{\sqrt{M^2 + \mathbf{P}^2}}$$

$$[D, Q_i] = -\mathrm{i}\,(\mathrm{e}^{-\tau P_0} Q_i + Q_i\,\mathrm{e}^{-\tau P_0}) \qquad [Q_i, P_j] = \mathrm{i}\,\delta_{ij}(1 + \tau\sqrt{M^2 + \mathbf{P}^2}) \tag{5.4}$$

$$[Q_i, Q_j] = -\mathrm{i}\,\varepsilon_{ijk}\left(\frac{\Sigma_k}{M^2 + \mathbf{P}^2} + \frac{\tau^2 J_k}{\tanh(\tau P_0/2)}\right).$$

Therefore, the position operators are transformed as a classical vector under rotations, but as a deformed one under dilations. The velocity for massless particles reduces to $V = 1$ as in special relativity. In this sense we remark that, very recently, this result has been obtained for all known DSR theories in [41], including κ-Poincaré, by using a Hamiltonian approach.

Furthermore, if we consider a spinless massive particle, the kinematical observables vanish so that position and momentum operators verify

$$[Q_i, Q_j] = -\mathrm{i}\,\varepsilon_{ijk}\frac{\tau^2 J_k}{\tanh(\tau P_0/2)} \qquad [Q_i, P_j] = \mathrm{i}\,\delta_{ij}\,\mathrm{e}^{\tau P_0}. \tag{5.5}$$

The latter commutation rule leads to the following generalized uncertainty principle:

$$\Delta Q_i \Delta P_j \geqslant \tfrac{1}{2}\delta_{ij}\langle \mathrm{e}^{\tau P_0}\rangle = \tfrac{1}{2}\delta_{ij}\langle 1 + \tau\sqrt{M^2 + \mathbf{P}^2}\rangle \tag{5.6}$$

where $\langle\cdot\rangle$ is the expectation value and Δ here means a root-mean-square deviation. We stress that, by following the arguments proposed in [43], expression (5.6) can be interpreted as a linear correction in ΔP of the usual Heisenberg uncertainty relation, whilst the κ-Poincaré construction leads to a quadratic term in ΔP. In this respect, see [1] for a comprehensive discussion of deformed uncertainty relations arising in quantum gravity theories.

6. Concluding remarks

We have presented a first example of a deformed relativistic theory based on a quantum group symmetry larger than Poincaré: the Weyl–Poincaré algebra. We also expect that the very same approach may be applied to other quantum deformations of $\mathcal{WP}$ as well as to quantum $so(4, 2)$ algebras. A first possibility is the so-called length-like (or space-type) deformation $U_\sigma(\mathcal{WP}) \subset U_\sigma(so(4, 2))$ [25], for which the deformation parameter has dimensions of length, which has been shown to be the symmetry algebra of a space discretization of the Minkowskian spacetime in one spatial direction. In the $(1 + 1)$-dimensional case [23], there is indeed an algebraic duality that relates both types of deformations $U_\tau(so(2, 2)) \leftrightarrow U_\sigma(so(2, 2))$ as well as $U_\tau(\mathcal{WP}_{1+1}) \leftrightarrow U_\sigma(\mathcal{WP}_{1+1})$, by interchanging the energy P_0 and the momentum P_1. Nevertheless, this equivalence is 'broken' in 3 + 1 dimensions in such a manner that a single 'privileged' discrete space direction arises. Thus, isotropy of the space is removed and this fact may preclude further possible physical implications. Another possibility worth studying is the twisted conformal algebra $U_z(so(4, 2))$ [27] which also contains a deformed

$\mathcal{WP}$ subalgebra; this case should provide a DSR theory naturally adapted to a null-plane (light-cone) framework, since the deformation parameter z can be interpreted in a natural way as the lattice step along two null-plane directions.

To end with, we would like to point out that in order to complete the DSR theory provided by $U_\tau(\mathcal{WP})$, the corresponding (dual) quantum group should be explicitly computed. This would give rise to an associated non-commutative Minkowskian spacetime which, by taking into account the results of section 4, should be different from the well-known κ-Minkowski spacetime [20, 34, 44, 45]. Work along this line is in progress.

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